Recent developments on log-concavity and q-log-concavity of combinatorial polynomials

William Y.C. Chen

joint work with

Cindy C.Y. Gu, Sabrina X.M. Pang, Ellen X.Y. Qu, Robert L. Tang, Carol J. Wang, Larry X.W. Wang, Ernest X.W. Xia, Arthur L.B. Yang

Center for Combinatorics
Nankai University

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Definitions

Let $\{a_i\}_{0\leq i\leq m}$ be a positive sequence of real numbers.

Definition

$\{a_i\}_{0\leq i\leq m}$ is unimodal if there exists k such that

$$a_0 \leq \cdots \leq a_k \geq \cdots \geq a_m,$$

and is strictly unimodal if

$$a_0 < \cdots < a_k > \cdots > a_m.$$

Example

For fixed m, $\{\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}\}$ is symmetric and unimodal. Furthermore, it is strictly unimodal if m is even.

Definitions

Definition

$\{a_i\}_{0 \le i \le m}$ *is* *log-concave* *if*

$$a_i^2 \geq a_{i+1} a_{i-1}$$

for all $1 \le i \le m-1$, *and is* *strictly log-concave* *if*

$$a_i^2 > a_{i+1} a_{i-1}.$$

$f(x)$

Remark: A log-concave sequence is unimodal.

Example

For fixed m, $\{\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}\}$ *is strictly log-concave. While* $\{1, 3, 5, 9, 5, 3, 1\}$ *is unimodal, but not log-concave.*

Definitions

Let $f(q) = a_0 + a_1 q + \cdots + a_m q^m$ be a polynomial with real coefficients.

Definition

$f(q)$ is unimodal (or strictly unimodal) if $\{a_i\}_{0 \le i \le m}$ is unimodal (resp. strictly unimodal).

Definition

$f(q)$ is log-concave (or strictly log-concave) if $\{a_i\}_{0 \le i \le m}$ is log-concave (resp. strictly log-concave).

Example

Let $des(\pi)$ denote the number of descents of π. The Eulerian polynomial $A_m(q) = \sum_{\pi \in \mathfrak{S}_m} q^{1+des(\pi)}$ is strictly log-concave.

Definitions

Let $\{f_i(q)\}_{0\le i\le m}$ be a sequence of polynomials with real coefficients.

Definition

For any two polynomials $f(q)$ and $g(q)$ with real coefficients, define $f(q) \ge_q g(q)$ if and only if $f(q) - g(q)$, as a polynomial in q, has all nonnegative coefficients.

Definition

$\{f_i(q)\}_{0\le i\le m}$ *is q-log-concave if*

$$f_i(q)^2 \ge_q f_{i+1}(q)f_{i-1}(q), \quad 1 \le i \le m-1,$$

and is strongly q-log-concave if

$$f_i(q)f_j(q) \ge_q f_{i+1}(q)f_{j-1}(q), \quad i \ge j \ge 1.$$

Definitions

Example

The Gaussian binomial coefficients $\{\left[{m \atop k} \right]_q\}_{0\le k\le m}$ *are strongly q-log-concave.*

- *The q-log-concavity was conjectured by Butler (1987).*
- *The first proof was given by Butler (1990).*
- *Krattenthaler (1989) found an alternative combinatorial proof.*
- *Sagan (1992) gave an inductive proof.*

Remark: Usually, a q-log-concave sequence is not strongly q-log-concave.

Example

The sequence $\{q^2, q+q^2, 1+2q+q^2, 4+q+q^2\}$ *is q-log concave but not strongly q-log concave.*

Definitions

Based on the q-log-concavity, it is natural to define the q-log-convexity.

Definition

$\{f_i(q)\}_{0\le i\le m}$ is q-log-convex if

$$f_i(q)^2 \le_q f_{i+1}(q)f_{i-1}(q), \quad 1 \le i \le m-1,$$

and is strongly q-log-convex if

$$f_i(q)f_j(q) \le_q f_{i+1}(q)f_{j-1}(q), \quad i \ge j \ge 1.$$

Example

The sequence $\{2q+q^2+3q^3, q+2q^2+2q^3, q+2q^2+2q^3, 2q+q^2+3q^3\}$ is q-log-convex, but not strongly q-log-convex.

Overview

I wish to report the following work on log-concavity and q-log-concavity of combinatorial polynomials.

(1) the unimodality conjecture of Palmer, Read and Robinson on the number of balanced coloring of the n-cube; a log-concavity theorem for sufficiently large n.

(2) the ratio monotonicity, reverse ultra log-concavity and 2-log-concavity of the Boros-Moll polynomials; the combinatorial proof of log-concavity of Boros-Moll polynomials; the 2-log-convexity of Apéry numbers;

(3) a symmetric function approach to the q-log-convexity conjectures, due to Liu and Wang, on the Narayana polynomials of type A and type B;

Overview

(continued)

(4) the strong log-concavity of q-Narayana numbers and a conjecture of McNamara and Sagan on the infinite q-log-concavity of the Gaussian coefficients;

(5) a unified approach to the q-log-convexity of the Bell polynomials, the Bessel polynomials, the Ramanujan polynomials and the Dowling polynomials, based on a triangular recurrence relation.

(6) Some open problems on log-concavity and q-log-concavity of polynomials.

Outline

1. *Balanced Colorings of n-Cube*
2. *Boros-Moll polynomials*
3. *q-Narayana Numbers*
4. *Narayana polynomials*
5. *A Class of Strongly q-Log-convex Polynomials*
6. *Some Open Problems*

Background

Let Q_n be the n-dimensional cube represented by a graph whose vertices are sequences of 1's and -1's of length n, where two vertices are adjacent if they differ only at one position.

Let V_n denote the set of vertices of Q_n, namely,

$$V_n = \{(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \,|\, \epsilon_i = -1 \text{ or } 1,\ 1 \leq i \leq n\}.$$

By a 2-coloring of the Q_n we mean an assignment of weights 1 or 0 to the vertices of Q_n.

Background

The weight of a 2-coloring is the sum of weights or the numbers of vertices with weight 1.
The center of mass of a coloring f with $w(f) \neq 0$ is the point whose coordinates are given by

$$\frac{1}{w(f)} \sum (\epsilon_1, \epsilon_2, \ldots, \epsilon_n),$$

where the sum ranges over all black vertices. If $w(f) = 0$, we take the center of mass to be the origin.

Background

A 2-coloring is balanced if its center of mass coincides with the origin. A pair of vertices of the n-cube is called an antipodal pair if it is of the form $(v, -v)$. A 2-coloring is said to be antipodal if any vertex v and its antipodal have the same color.

Let $\mathcal{B}_{n,2k}$ denote the set of balanced 2-colorings of the n-cube with exactly $2k$ black vertices and $B_{n,2k} = |\mathcal{B}_{n,2k}|$.

Conjecture (Palmer-Read-Robinson, J. Algebraic Combin. (1992))

The sequence $\{B_{n,2k}\}_{0 \le k \le 2^{n-1}}$ is unimodal with the maximum at $k = 2^{n-2}$ for any $n \geq 1$.

Refinement of Balanced Colorings

Example: When $n = 4$, the sequence $\{B_{n,2k}\}$ reads

$$1, 8, 52, 152, 222, 152, 52, 8, 1,$$

which is a unimodal sequence.
Let $\mathcal{B}_{n,2k,i}$ denote the set of the balanced 2-colorings in $\mathcal{B}_{n,2k}$ containing exactly i antipodal pairs of black vertices.

Theorem (Chen-Wang, J. Algebraic Combin. (2010))

For $0 \le i \le k$ and $0 \le k \le 2^{n-2} - 1$, we have

$$(2^{n-1} - 2k + i)|\mathcal{B}_{n,2k,i}| = (i+1)|\mathcal{B}_{n,2k+2,i+1}|. \tag{1}$$

Proof of the Palmer-Read-Robinson Conjecture

This theorem implies that $|\mathcal{B}_{n,2k,i}| < |\mathcal{B}_{n,2k+2,i+1}|$ for $0 \leq k \leq 2^{n-2} - 1$. Thus we have

$$B_{n,2k} = \sum_{i=0}^{k} |\mathcal{B}_{n,2k,i}| < \sum_{i=1}^{k+1} |\mathcal{B}_{n,2k+2,i}| \leq \sum_{i=0}^{k+1} |\mathcal{B}_{n,2k+2,i}| = B_{n,2k+2},$$

for $0 \leq k \leq 2^{n-2} - 1$. Since $\{B_{n,2k}\}_{0 \leq k \leq 2^{n-1}}$ is symmetric for any $n \geq 1$, the Palmer-Read-Robinson Conjecture is true.

Log-concavity Conjecture

The sequence $\{B_{n,2k}\}_{0\leq k\leq 2^{n-1}}$ not log-concave in general.
Example: $B_{5,0}=1$, $B_{5,2}=16$ and $B_{5,4}=320$, we have
$B_{5,2}^2-B_{5,0}B_{5,4}<0$.
However, we observed that $\{B_{n,2k}\}_n$ is log-concave for small k.

Conjecture (Chen-Wang, J. Algebraic Combin. (2010))

When $0\leq k\leq 2^{n-1}$, we have

$$B_{n,2k}^2\geq B_{n-1,2k}B_{n+1,2k}.$$

Applying the probabilistic method, we shall show that this conjecture holds for sufficiently large n.

Probabilistic Method

Theorem (Canfield-Gao-Greenhill-McKay-Robinson, 2009, arXiv)

If $0 \le k \le o(2^{n/2})$, *then*

$$B_{n,2k} = \binom{2k}{k}^n \left(1 - O\left(\frac{k^2}{2^n}\right)\right) / (2k)!.$$

Theorem (Chen-Wang, J. Algebraic Combin. (2010))

Let $c_{n,k}$ *be the real number such that*

$$B_{n,2k} = \binom{2k}{k}^n \left(1 - c_{n,k}\left(\frac{k^2}{2^n}\right)\right) / (2k)!. \qquad (2)$$

Then, for $k \ge 3$ *and* $n > 5\log_{\frac{4}{3}} k + \log_{\frac{4}{3}} 96$, *we have* $c_{n,k} > c_{n+1,k}$.

Probabilistic Method

Applying the theorem above, we arrive at the following result by a direct calculation.

Theorem (Chen-Wang, J. Algebraic Combin. (2010))

When $n \geq 5 \log_{\frac{4}{3}} k + \log_{\frac{4}{3}} 96$, *we have*

$$B_{n,2k}^2 > B_{n-1,2k} B_{n+1,2k}.$$

Outline

Boros-Moll Polynomials

Boros and Moll explored the following quartic integral.

Theorem (Moll, Notices Amer. Math. Soc. (2002))

For any $a > -1$ and any nonnegative integer m,

$$\int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}}\,dx = \frac{\pi P_m(a)}{2^{m+3/2}(a+1)^{m+1/2}},$$

where

$$P_m(a) = \sum_{j,k} \binom{2m+1}{2j}\binom{m-j}{k}\binom{2k+2j}{k+j}\frac{(a+1)^j(a-1)^k}{2^{3(k+j)}}.$$

Proof. It follows from Wallis's integral formula.
The polynomials $P_m(a)$ will be called the Boros-Moll polynomials.

Boros-Moll Polynomials

It is not clear that the polynomial $P_m(a)$ has positive coefficients from the above double summation formula.
Ramanujan's Master Theorem yields the following formula.

Theorem (Moll, Notices Amer. Math. Soc. (2002))

For any m,

$$P_m(a) = 2^{-2m} \sum_k 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} (a+1)^k. \qquad (3)$$

Boros-Moll Polynomials

Let $d_i(m)$ be given by $P_m(a) = \sum_{i=0}^{m} d_i(m)a^i$.

From (3), it follows that

$$d_i(m) = 2^{-2m} \sum_{k=i}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} \binom{k}{i}. \tag{4}$$

Unimodality and Log-concavity

Theorem (Boros and Moll, J. Math. Anal. Appl. (1999))

The sequence $\{d_i(m)\}_{0\le i\le m}$ is unimodal and the maximum element appears in the middle. In other words,

$$d_0(m) < \cdots < d_{[\frac{m}{2}]}(m) > d_{[\frac{m}{2}]+1}(m) > \cdots > d_m(m).$$

Theorem (Kauers and Paule, Proc. Amer. Math. Soc. (2007))

The sequence $\{d_i(m)\}_{0\le i\le m}$ is log-concave.

Remark. This was conjectured by Moll (2002). Proof is based on recurrence relations obtained by symbolic computations.

Recurrences of Kauers and Paule

Kauers and Paule (2007) utilized the RISC package MultiSum to derive the following recurrences: for $0 \leq i \leq m$

$$d_i(m+1) = \frac{m+i}{m+1} d_{i-1}(m) + \frac{(4m+2i+3)}{2(m+1)} d_i(m), \quad 0 \leq i \leq m+1, \tag{5}$$

$$d_i(m+1) = \frac{(4m-2i+3)(m+i+1)}{2(m+1)(m+1-i)} d_i(m) - \frac{i(i+1)d_{i+1}(m)}{(m+1)(m+1-i)}, \tag{6}$$

$$d_i(m+2) = \frac{-4i^2+8m^2+24m+19}{2(m+2-i)(m+2)} d_i(m+1) - \frac{(m+i+1)(4m+3)(4m+5)}{4(m+2-i)(m+1)(m+2)} d_i(m),, \tag{7}$$

$$d_{i-2}(m) = \frac{(i-1)(2m+1)d_{i-1}(m)}{(m+2-i)(m+i-1)} - \frac{i(i-1)d_i(m)}{(m+2-i)(m+i-1)}. \tag{8}$$

The Ratio Monotone Property

A sequence $\{a_i\}_{0\leq i\leq m}$ of positive numbers is said to be spiral if

$$a_m \leq a_0 \leq a_{m-1} \leq a_1 \leq \cdots \leq a_{[\frac{m}{2}]}.$$

A sequence $\{a_i\}_{0\leq i\leq m}$ of positive numbers is said to be ratio monotone if

$$\frac{a_0}{a_{m-1}} \leq \frac{a_1}{a_{m-2}} \leq \cdots \leq \frac{a_{i-1}}{a_{m-i}} \leq \frac{a_i}{a_{m-1-i}} \leq \cdots \leq \frac{a_{[\frac{m}{2}]-1}}{a_{m-[\frac{m}{2}]}} \leq 1,$$

$$\frac{a_m}{a_0} \leq \frac{a_{m-1}}{a_1} \leq \cdots \leq \frac{a_{m-i}}{a_i} \leq \frac{a_{m-1-i}}{a_{i+1}} \leq \cdots \leq \frac{a_{m-[\frac{m-1}{2}]}}{a_{[\frac{m-1}{2}]}} \leq 1.$$

If the above inequalities become strict, we say that the sequence is strictly ratio monotone. It is easy to see that the ratio monotonicity implies log-concavity and spiral property.

The Ratio Monotone Property

Theorem (Chen-Xia, Math. Comput. (2009))

The Boros-Moll sequence $\{d_i(m)\}_{0\leq i\leq m}$ satisfies the strictly ratio monotone property.

Proof. We mainly use the four recurrence relations given by Kauers and Paule and the following lower and upper bounds of $d_i(m+1)/d_i(m)$.

Lower Bound

Theorem (Chen-Xia, Math. Comput. (2009))

Let $m \geq 2$. We have for $1 \leq i \leq m-1$,

$$d_i(m+1) > \frac{4m^2+7m+i+3}{2(m+1-i)(m+1)} d_i(m), \tag{9}$$

and

$$d_0(m+1) = \frac{4m+3}{2(m+1)} d_0(m), \tag{10}$$

$$d_m(m+1) = \frac{(2m+3)(2m+1)}{2(m+1)} 2^{-m} \binom{2m}{m}. \tag{11}$$

Upper Bound

Theorem (Chen-Xia, Math. Comput. (2009))

Let $m \geq 2$ be a positive integer. We have for $0 \leq i \leq m$,

$$d_i(m+1) \leq B(m,i)d_i(m), \tag{12}$$

where $B(m,i)$ is defined by

$$B(m,i) = \frac{A(m,i)}{2(i+2)(4m+2i+5)(m+1)(m-i+1)} \tag{13}$$

with

$$\begin{aligned} A(m,i) =& 30 + 96m^2 + 94m + 37i + 72m^2 i + 8m^2 i^2 - i^3 \\ &+ 99mi + 5i^2 + 13mi^2 + 16m^3 i + 32m^3. \end{aligned} \tag{14}$$

Ultra Log-Concavity

A positive sequence $\{a_k\}_{0\le k\le n}$ is ultra log-concave if $\left\{a_k/\binom{n}{k}\right\}$ is log-concave. This condition can be restated as

$$(n-k)ka_k^2-(n-k+1)(k+1)a_{k-1}a_{k+1}\ge 0.$$

Newton's inequality: if the polynomial $\sum_{k\ge 0}^n a_kx^k$ with positive coefficients has only real zeros, then the sequence $a_0,a_1,\ldots,a_n$ is ultra log-concave.

Reverse Ultra Log-Concavity

A positive sequence $\{a_k\}_{0\leq k\leq n}$ is said to be reverse ultra log-concave if it satisfies the reverse relation of ultra log-concavity, that is,

$$k(n-k)a_k^2-(n-k+1)(k+1)a_{k-1}a_{k+1}\leq 0.$$

Example

For $n\geq 2$, the Bessel polynomial

$$y_n(x)=\sum_{k=0}^{n}\frac{(n+k)!}{2^k k!(n-k)!}x^k$$

is log-concave and reverse ultra log-concave.

Reverse Ultra Log-Concavity

Theorem (Chen-Gu, Proc. Amer. Math. Soc. (2009))

For all $m \geq 2$, $1 \leq i \leq m-1$, we have $\frac{d_i(m+1)}{d_i(m)} < T(m,i)$, where

$$T(m,i) = \frac{4m^2 + 7m + 3 + i\sqrt{4m + 4i^2 + 1} - 2i^2}{2(m-i+1)(m+1)},$$

and for $m \geq 1$, we have

$$\frac{d_0(m+1)}{d_0(m)} = T(m,0), \qquad \frac{d_m(m+1)}{d_m(m)} = T(m,m).$$

Theorem (Chen-Gu, Proc. Amer. Math. Soc. (2009))

The Boros-Moll sequence $\{d_i(m)\}_{0 \leq i \leq m}$ satisfies the reverse ultra log-concave property.

A Lower Bound for $d_i(m)^2/(d_{i-1}(m)d_{i+1}(m))$

On the other hand, the coefficients $d_i(m)$ satisfy an inequality stronger than the log-concavity.

Theorem (Chen-Gu, Proc. Amer. Math. Soc. (2009))

For $m \geq 2$ and $1 \leq i \leq m-1$, we have

$$\frac{d_i(m)^2}{d_{i-1}(m)d_{i+1}(m)} > \frac{(m-i+1)(i+1)(m+i)}{(m-i)i(m+i+1)}.$$

Corollary (Chen-Gu, Proc. Amer. Math. Soc. (2009))

The sequence $\{i!d_i(m)\}$ is log-concave.

Moll's Minimum Conjecture

Theorem (Chen-Xia, European J. Combin. (2010))

For $1 \leq i \leq m$,

$$i(i+1)\left(d_i^2(m) - d_{i+1}(m)d_{i-1}(m)\right)$$

attains its minimum at $i = m$ with $2^{-2m}m(m+1)\binom{2m}{m}^2$.

This was conjectured by Moll (2005).
Proof is based on the log-concavity of $\{i!d_i(m)\}$ and the ratio monotone property of $\{d_i(m)\}$.

A Lower Bound for $d_i(m)^2/(d_{i-1}(m)d_{i+1}(m))$

Since

$$\frac{(m-i+1)(i+1)(m+i)}{(m-i)i(m+i+1)} < \frac{d_i(m)^2}{d_{i-1}(m)d_{i+1}(m)} < \frac{(m-i+1)(i+1)}{(m-i)i},$$

we have

Corollary (Chen-Gu, Proc. Amer. Math. Soc. (2009))

For $1 \le i \le m-1$, let

$$c_i(m) = \frac{d_i^2(m)}{d_{i-1}(m)d_{i+1}(m)} \quad , \quad u_i(m) = \left(1+\frac{1}{i}\right)\left(1+\frac{1}{m-i}\right).$$

Then for any $i \ge 1$,

$$\lim_{m\to\infty} \frac{c_i(m)}{u_i(m)} = 1.$$

Interlacing Log-Concvaity

Note that log-concavity and ratio monotone property are both the relations of ratio of one row. We call a sequence $\{a_i(m)\}$ is interlacing log-concave if

$$\begin{aligned} & r_0(m+1) \leq r_0(m) \leq r_1(m+1) \\ & \quad \leq r_1(m) \leq \cdots \leq r_{m-1}(m+1) \leq r_{m-1}(m) \leq r_m(m+1), \end{aligned}$$

where

$$r_i(m) = a_i(m)/a_{i+1}(m).$$

Interlacing Log-Concvaity

We found the Boros-Moll polynomials process the interlacing log-concavity. For example, for $n = 4, 5$, we have

$$P_4(a) = \frac{1155}{128} + \frac{885}{32}a + \frac{1095}{32}a^2 + \frac{315}{16}a^3 + \frac{35}{8}a^4,$$

$$P_5(a) = \frac{4389}{256} + \frac{8589}{128}a + \frac{7161}{64}a^2 + \frac{777}{8}a^3 + \frac{693}{16}a^4 + \frac{63}{8}a^5.$$

The interlacing log-concavity is illustrated as follows:

$$\frac{\frac{4389}{256}}{\frac{8589}{128}} < \frac{\frac{1155}{128}}{\frac{885}{32}} < \frac{\frac{8589}{128}}{\frac{7161}{64}} < \frac{\frac{885}{32}}{\frac{1095}{32}} < \frac{\frac{7161}{64}}{\frac{777}{8}} < \frac{\frac{1095}{32}}{\frac{315}{16}} < \frac{\frac{777}{8}}{\frac{693}{16}} < \frac{\frac{315}{16}}{\frac{35}{8}} < \frac{\frac{693}{16}}{\frac{63}{8}}.$$

Interlacing Log-Concavity

By induction, we obtain the following two lemmas from which we establish the interlacing log-concavity of the Boros-Mill polynomials

Lemma (Chen-Wang-Xia, preprint)

Let $m \geq 2$ be an integer. For $0 \leq i \leq m-2$, we have

$$\frac{d_i(m)}{d_{i+1}(m)} < \frac{4m+2i+3}{4m+2i+7}\frac{d_{i+1}(m)}{d_{i+2}(m)}.$$

Lemma (Chen-Wang-Xia, 2010, arXiv)

Let $m \geq 2$ be a positive integer. For $0 \leq i \leq m-1$, we have

$$\frac{d_i(m)}{d_{i+1}(m)} > \frac{2i+4m+5}{2i+4m+3}\frac{d_i(m+1)}{d_{i+1}(m+1)}.$$

Interlacing Log-Concavity

Theorem (Chen-Wang-Xia, 2010, arXiv)

Suppose the triangular array $T(n,k) > 0$ satisfies the recurrence

$$T(n,k) = f(n,k)T(n-1,k) + g(n,k)T(n-1,k-1),$$

and the polynomial $\sum_{k=0}^{n} T(n,k)x^k$ has only real zeros for every n. If

$$\frac{(n-k)k}{(n-k+1)(k+1)} f(n+1,k+1) \le f(n+1,k) \le f(n+1,k+1)$$

and

$$g(n+1,k+1) \le g(n+1,k) \le \frac{(n-k+1)(k+1)}{(n-k)k} g(n+1,k+1),$$

then the triangular array $T(n,k)$ satisfies the interlacing log-concavity.

Interlacing Log-Concavity

This result applies to the following sequences.

(1) The stirling numbers of the first kind, $c(n,k)$ with the recurrence

$$c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1).$$

(2) The Stirling numbers of the second kind, S(n,k) with the recurrence

$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

(3) The holiday numbers $\psi(n,k)$ and $\phi(n,k)$ of the first kind and the second kind with the recurrences

$$\psi(n,k) = (2n+k-1)\psi(n-1,k) + \psi(n-1,k-1)$$

and

$$\phi(n,k) = (2n+k)\phi(n-1,k) + \phi(n-1,k-1),$$

respectively.

(4) The Whitney numbers $W_m(n,k)$ satisfies the recurrence

$$W_m(n,k) = (1+mk)W_m(n-1,k) + W_m(n-1,k-1).$$

Combinatorics of Boros-Moll polynomials

From the combinatorial point of view, it is always interesting to find combinatorial reasons for the properties of the Boros-Moll polynomials such as positivity, unimodality and log-concavity. We have explained the positivity combinatorially. It is also desirable to find combinatorial proofs of unimodal and log-concave properties. Furthermore, it would be interesting to find combinatorial interpretations of the recurrence relations of $d_i(m)$.

Positivity of Boros-Moll polynomials

Chen-Pang-Qu (Ramanujan J. (2010)) gave a Combinatorial proof for the equivalence of the following two expressions:

$$P_m(a) = \sum_{j,k} \binom{2m+1}{2j}\binom{m-j}{k}\binom{2k+2j}{k+j}\frac{(a+1)^j(a-1)^k}{2^{3(k+j)}},$$

$$P_m(a) = 2^{-2m}\sum_{k} 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}(a+1)^k.$$

Tools: reluctant functions & an extension of Foata's bijection

Log-concavity of Boros-Moll polynomials

Chen-Pang-Qu (preprint) found a combinatorial proof of the log-concavity of $\{d_i(m)\}_{0 \leq i \leq m}$:
First, rewrite

$$d_i(m)^2 \geq d_{i-1}(m)d_{i+1}(m)$$

as follows:

$$\begin{aligned}(m+i+1)D_{i+1}(m) \cdot (m-i+1)D_{i-1}(m) \leq (m+i)(m-i+1)D_i^2(m) \\ +\tfrac{1}{i}(m+i)(m-i)D_i^2(m) + \tfrac{1}{i}(m+i)D_i^2(m), \quad (15)\end{aligned}$$

where $D_i(m) := \binom{2m}{m-i} m! i! (m-i)! 2^i d_i(m)$.

Combinatorial Interpretation of $D_i(m)$

By straightforward calculation,

$$\begin{aligned} D_i(m) &= \binom{2m}{m-i} m!i!(m-i)!2^i d_i(m) \\ &= \binom{2m}{m-i} \sum_{\ell=0}^{m-i} \binom{m-i}{\ell} \left(\frac{1}{2}\right)^{\ell} \left(\frac{1}{2}\right)_{m-i-\ell} (1)_{m+i+\ell} \qquad (16) \end{aligned}$$

where $(x)_n := x(x+1)\cdots(x+n-1)$. Note that$(x)_n$ equals the generating function of permutations on $[n]$ with respect to the number of cycles.

Combinatorial Interpretation of $D_i(m)$

Suppose (A, B, C) is a composition of $[2m]$, namely, any two sets of A, B, C are disjoint and $A \cup B \cup C = [2m]$, where A, B and C are allowed to be empty. A pair of permutations $(\sigma_{AB}; \sigma_C)$ on $A \cup B$ and C respectively is called a *3-colored permutation* on $[2m]$. For example,

$$(2, 12, \mathbf{8}, 11, 5, 9, 7, \mathbf{1}, 4, 3; (6, 10))$$

is a 3-colored permutation, where the elements belonging to A are in boldface, σ_{AB} is expressed with one-line representation and σ_C is expressed by the canonical cycle representation.

Combinatorial Interpretation of $D_i(m)$

Assign the weight of a 3-colored permutation $(\sigma_{AB};\sigma_C)$ by the following rules:

- The weight of an element in A, B, C is given by $\frac{1}{2}, 1, 1$, respectively;
- The weight of a cycle in σ_{AB} is given by 1;
- The weight of a cycle in σ_C is given by $\frac{1}{2}$.

Let $\mathcal{D}_i(m)$ denote the set of all 3-colored permutations $(\sigma_{AB};\sigma_C)$ on $[2m]$ such that the cardinality of B is $m+i$. Then by (16), $D_i(m)$ is the weight sum of 3-colored permutations in $\mathcal{D}_i(m)$.

Log-concavity of Boros-Moll polynomials

By this combinatorial interpretation of $D_i(m)$ we can give a combinatorial proof of the relation

$$(m+i+1)D_{i+1}(m)\cdot(m-i+1)D_{i-1}(m) < (m+i)(m-i+1)D_i^2(m). \quad (17)$$

This is achieved by two weight preserving correspondences.

Log-concavity of Boros-Moll polynomials

The same combinatorial approach can be also used to give a bijective proof of the the relation

$$\frac{1}{2}(m+i+1)D_{i+1}(m)+2(m-i+1)D_{i-1}(m)=(2m+1)D_i(m),$$

which is equivalent to the recurrence relation

$$i(i+1)d_{i+1}(m)=i(2m+1)d_i(m)-(m-i+1)(m+i)d_{i-1}(m) \quad (18)$$

given by Kauers and Paule, and Moll independently.

2-Log-Concavity

Define the $\mathcal{L}$-operator on sequences to be $\mathfrak{L}(a_k) = a_k^2 - a_{k-1}a_{k+1}$.
A sequence $\{a_k\}$ is *i*-fold log-concave if $\mathfrak{L}^j(a_k)$ is log-concave for $1 \leq j \leq i-1$.
If $\{a_k\}$ is *i*-fold log-concave for any i, then it is said to be ∞-log-concave.

Conjecture (Moll, Notices Amer. Math. Soc. (2002))

The sequence $d_i(m)$ is ∞-log-concave.

Conjecture (Brändén, 2009, arXiv)

Let $Q(x) = \sum_{i=0}^{m} \frac{d_i(m)}{i!} x^i$ and $R(x) = \sum_{i=0}^{m} \frac{d_i(m)}{(i+2)!} x^i$.
Then both $Q(x)$ and $R(x)$ have only real zeros.

Remark. The real-rootedness of $Q(x)$ (resp. $R(x)$) leads to the 2-fold (resp. 3-fold) log-concavity of $d_i(m)$.

2-Log-concavity

Kauers and Paule (Proc. Amer. Math. Soc., 2007) considered the 2-log-concavity of Boros-Moll sequences, they said "we have tried to apply the proof technique of Section 3 to establish 2-log-concavity, i.e.,

$$\left(d_l^2(m) - d_{l-1}(m)d_{l+1}(m)\right)^2 \\ - \left(d_{l-1}^2(m) - d_{l-2}(m)d_l(m)\right)\left(d_{l+1}^2(m) - d_{l+2}(m)d_l(m)\right) > 0.$$

The recurrences (9) and (6) can again be used for obtaining an equivalent statement involving only shifts in m but no shifts in l. This statement is polynomial in the $d_l(m+i)$ of degree 4. As a consequence, the condition corresponding to (13) is much more complicated. It involves algebraic functions of degree up to 15, and it would require more than thirty pages to print it here. Under these circumstances, we have little hope that a proof of 2-log-concavity could be completed along these lines. "

2-Log-Concavity

Theorem (Chen-Xia, preprint (2010))

The sequence $d_i(m)$ is 2-log-concave.

The key idea is to find a function $f(m,k)$ such that

$$\frac{d_{i-1}^2(m) - d_{i-2}(m)d_i(m)}{d_i^2(m) - d_{i-1}(m)d_{i+1}(m)} < f(m,i) < \frac{d_i^2(m) - d_{i-1}(m)d_{i+1}(m)}{d_{i+1}^2(m) - d_i(m)d_{i+2}(m)},$$

where

$$f(m,i) = \frac{(i+1)(i+2)(m+i+3)^2}{(m+1-i)(m+2-i)(m+i+2)^2}.$$

2-Log-Concavity

Theorem (Brändén, arXiv, 2009)

If the polynomial

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

has real and negative zeros, then the sequence $\{a_k\}_{k=0}^n$ is ∞-log-concave.

This was conjectured independently by Stanley, McNamara-Sagan and Fisk.

Corollary

The binomial coefficients $\{\binom{n}{k}\}_k$ is ∞-log-concave.

This was first conjectured by Boros and Moll.

2-Log-Convexity of Apéry Numbers

In his proof of the irrationality of $\zeta(2)$ and $\zeta(3)$, Apéry introduced the following numbers A_n and B_n as given by

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \tag{19}$$

$$B_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}. \tag{20}$$

The numbers A_n and B_n are often called the Apéry numbers.

2-*Log-Convexity of Apéry Numbers*

It has been shown by Apéry that A_n and B_n satisfy the following three-term recurrence relations for $n \geq 2$,

$$A_n = \frac{34n^3 - 51n^2 + 27n - 5}{n^3} A_{n-1} - \frac{(n-1)^3}{n^3} A_{n-2},$$

$$B_n = \frac{11n^2 - 11n + 3}{n^2} B_{n-1} + \frac{(n-1)^2}{n^2} B_{n-2},$$

where $A_0 = 1,\ A_1 = 5,\ B_0 = 1,\ B_1 = 3;$.

2-Log-Convexity of Apéry Numbers

Cohen and Rhin obtained the following recurrence relation of U_n in connection with the rational approximation of $\zeta(4)$,

$$U_{n+1} = R(n)U_n + G(n)U_{n-1}, \qquad n \geq 1,$$

where $U_0 = 1$, $U_1 = 12$ and

$$R(n) = \frac{3(2n+1)(3n^2+3n+1)(15n^2+15n+4)}{(n+1)^5},$$

$$G(n) = \frac{3n^3(3n-1)(3n+1)}{(n+1)^5}.$$

.

2-Log-Convexity of Apéry Numbers

Let

$$\begin{aligned} a_3(n) =& 2b(n+2)b^2(n+1) + 2b(n+1)c(n+2) - b^3(n+1) \\ & - b(n+1)b(n+2)b(n+3) - b(n+3)c(n+2) - c(n+3)b(n+1), \\ a_2(n) =& 4b(n+1)b(n+2)c(n+1) + 2c(n+1)c(n+2) + b(n+1)^2 b(n+2)b(n+3) \\ & + b(n+1)b(n+3)c(n+2) + b(n+1)^2 c(n+3) - 3c(n+1)b^2(n+1) \\ & - b(n+3)b(n+2)c(n+1) - c(n+3)c(n+1) - b^2(n+2)b^2(n+1) \\ & - 2b(n+2)b(n+1)c(n+2) - c^2(n+2), \\ a_1(n) =& -c(n+1)\big(2b(n+2)c(n+2) - 2b(n+2)c(n+1) - 2b(n+3)b(n+2)b(n+1) \\ & - b(n+3)c(n+2) - 2c(n+3)b(n+1) + 3c(n+1)b(n+1) + 2b^2(n+2)b(n+1)\big), \\ a_0(n) =& -c^2(n+1)\left(c(n+1) - b(n+2)b(n+3) - c(n+3) + b^2(n+2)\right) \end{aligned}$$

and

$$\Delta(n) = 4a_2^2(n) - 12a_1(n)a_3(n).$$

2-Log-Convexity of Apéry Numbers

Theorem (Chen-Xia, Proc. Amer. Math. Soc., to appear)

Suppose $\{S_n\}_{n=0}^{\infty}$ *is a positive log-convex sequence that satisfies the recurrence relation*

$$S(n) = b(n)S(n-1) + c(n)S(n-2) \qquad (21)$$

for $n \geq 2$. *Assume that* $a_3(n) < 0$ *and* $\Delta(n) > 0$ *for all* $n \geq N_0$, *where* N_0 *is a positive integer. If there exist* f_n *and* g_n *such that for all* $n \geq N_0$,

(C_1) $f_n \leq \frac{S_n}{S_{n-1}} < g_n$;

(C_2) $f_n \geq \frac{-2a_2(n)-\sqrt{\Delta(n)}}{6a_3(n)}$;

(C_3) $a_3(n)g_n^3 + a_2(n)g_n^2 + a_1(n)g_n + a_0(n) > 0$,

then $\{S_n\}_{n=N_0}^{\infty}$ *is strictly 2-log-convex.*

2-*Log-Convexity of Apéry Numbers*

This presents a unified approach for the 2-log-convexity of the Apéry numbers, the Cohen-Rhin numbers, the Motzkin numbers, the Fine numbers, the Franel numbers of order 3 and 4 and the large Schröder numbers.

Polynomials with nondecreasing and nonnegative coefficients

Let $P(x) = a_0 + a_1 x + \cdots + a_m x^m$, where $0 \leq a_0 \leq a_1 \leq \cdots \leq a_m$.

Theorem (Boros-Moll, Electron. J. Combin. (2001))

$P(x+1)$ is unimodal.

Theorem (Alvarez-Amadis-Boros-Karp-Moll-Rosales, Electron. J. Combin. (2001))

$P(x+n)$ is also unimodal for any positive integer n.

Theorem (Wang-Yeh, European J. Combin. (2005))

$P(x+c)$ is unimodal for any positive number c.

Polynomials with nondecreasing and nonnegative coefficients

Theorem (Llamas-Martínez-Bernal, 2010)

$P(x+c)$ is strictly log-concave for any $c \geq 1$.

Theorem (Chen-Yang-Zhou, 2010, arXiv)

$P(x+1)$ is ratio monotone.

Remark. The above results apply to the Boros-Moll polynomials $P_m(x)$.

$$P_m(x) = 2^{-2m} \sum_k 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} (x+1)^k.$$

Outline

q-Narayana Numbers

The q-Narayana numbers, as a natural q-analogue of the Narayana numbers $N(n,k)$, arise in the study of q-Catalan numbers. The q-Narayana number $N_q(n,k)$ is given by

$$N_q(n,k) = \frac{1}{[n]} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n \\ k-1 \end{bmatrix} q^{k^2-k},$$

where we have adopted the common notation

$$[k] := (1-q^k)/(1-q), \quad [k]! = [1][2]\cdots[k], \quad \begin{bmatrix} n \\ j \end{bmatrix} := \frac{[n]!}{[j]![n-j]!}$$

for the q-analogues of the integer k, the q-factorial, and the q-binomial coefficient, respectively.

Remark. q-Narayana Numbers have a symmetric function representation.

Partition

Given a nonnegative integer n, a partition λ of n is a weakly decreasing nonnegative integer sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{N}^k$ such that $\sum_{i=1}^{k} \lambda_i = n$.

The number of nonzero components λ_i is called the length of λ, denoted $\ell(\lambda)$. let $\mathrm{Par}(n)$ denote the set of all partitions of n.

Given two partitions λ and μ, we say $\mu \subseteq \lambda$, if $\lambda_i \geq \mu_i$ holds for each i.

Young Diagram

The Young diagram of λ is an array of squares in the plane justified from the top and left corner with $\ell(\lambda)$ rows and λ_i squares in row i.
When $\mu \subseteq \lambda$, a skew partition λ/μ is the diagram obtained from the diagram of λ by removing the diagram of μ at the top-left corner.

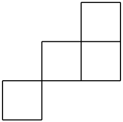

Fig 1: The diagram $(4, 3, 1)/(2, 1)$

Semistandard Young Tableau

A semistandard Young tableau (SSYT) of shape λ/μ is an array $T = (T_{ij})$ of positive integers of shape λ/μ that is weakly increasing in every row and strictly increasing in every column.
The type of T is defined as the composition $\alpha = (\alpha_1, \alpha_2, \ldots)$, where α_i is the number of i's in T.

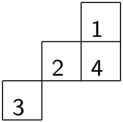

Fig 2: SSYT of shape $(4, 3, 1)/(2, 1)$

Schur Function

If T has type $\mathrm{type}(T) = \alpha$, then we write

$$x^T = x_1^{\alpha_1} x_2^{\alpha_2} \cdots .$$

The skew Schur function $s_{\lambda/\mu}(x)$ of shape λ/μ is defined as the generating function

$$s_{\lambda/\mu}(x) = \sum_T x^T,$$

summed over all semistandard Young tableaux T of shape λ/μ. We set $s_\emptyset(x) = 1$.
For a symmetric function $f(x)$, define

$$\begin{aligned} \mathrm{ps}_n(f) &= f(1, q, \ldots, q^{n-1}), \\ \mathrm{ps}_n^1(f) &= \mathrm{ps}_n(f)|_{q=1} = f(1^n). \end{aligned}$$

Hook-content Formula

A square (i,j) in λ is the square in row i from the top and column j from the left. The hook length $h(i,j)$, is given by $\lambda_i + \lambda'_j - i - j + 1$. The content $c(i,j)$ is given by $j - i$.

Theorem (Stanley, Studies in Applied Math. (1971))

For any partition λ and $n \geq 1$, we have

$$\begin{aligned} \mathrm{ps}_n(s_\lambda) &= q^{\sum_{k\geq 1}(k-1)\lambda_k} \prod_{(i,j)\in\lambda} \frac{[n+c(i,j)]}{[h(i,j)]} \\ \mathrm{ps}_n^1(s_\lambda) &= \prod_{(i,j)\in\lambda} \frac{n+c(i,j)}{h(i,j)}. \end{aligned}$$

Brändén's formula for q-Narayana Numbers

Brändén studied several Narayana statistics and bi-statistics on Dyck paths, and noticed that the q-Narayana number $N_q(n,k)$ has a Schur function expression by a specialization of the variables.

Theorem (Brändén, Discrete Math. (2004))

For all $n, k \in \mathbb{N}$, we have

$$N_q(n,k) = s_{(2^{k-1})}(q, q^2, \ldots, q^{n-1}). \tag{22}$$

Thus

$$N(n,k) = N_q(n,k)|_{q=1} = s_{(2^k)}(1^{n-1}) = \mathrm{ps}^1_{n-1} s_{(2^k)}.$$

q-Log-Concavity of $N_q(n,k)$ for fixed n

Theorem (Bergeron-McNamara, 2004, arXiv)

For $k \geq 1$ and $a \geq b$, the symmetric function $s_{(k^a)}s_{(k^b)} - s_{(k^{a+1})}s_{(k^{b-1})}$ is Schur positive.

The case of $a = b$ is due to Kirillov (1984), and a different proof was given by Kleber (2001).

Theorem (Chen-Wang-Yang, J. Algebraic Combin. (2010))

Given an integer n, the sequence $(N_q(n,k))_{k\geq 1}$ of polynomials in q is strongly q-log-concave.

q-Log-Concavity of $N_q(n,k)$ for fixed n

Proof. For any $k \geq l \geq 2$,

$$N_q(n,k)N_q(n,l) - N_q(n,k+1)N_q(n,l-1) = s_{(2^{k-1})}s_{(2^{l-1})} - s_{(2^k)}s_{(2^{l-2})},$$

where the Schur functions are evaluated at the variable set $\{q, q^2, \ldots, q^{n-1}\}$.
By Bergeron-McNamara's theorem, the difference $s_{(2^{k-1})}s_{(2^{l-1})} - s_{(2^k)}s_{(2^{l-2})}$ is Schur positive for $k \geq l$.
We see that the difference $N_q(n,k)N_q(n,l) - N_q(n,k+1)N_q(n,l-1)$ as a polynomial in q has nonnegative coefficients.

Transformation formulas

Employing the Hook-content formula, we can deduce the following relations used in the proof of the q-log-concavity of the q-Narayana numbers $N_q(n,k)$ for given k. For any $r \geq 1$, let

$$X_r = \{q, q^2, \ldots, q^{r-1}\}, \quad X_r^{-1} = \{q^{-1}, q^{-2}, \ldots, q^{-(r-1)}\}.$$

Lemma

For any $m \geq n \geq 1$ and $k \geq 1$, we have

$$\begin{aligned} & q^{n-1} s_{(2^{k-1},1)}(X_{n-1}) s_{(2^k)}(X_m) - q^m s_{(2^{k-1},1)}(X_m) s_{(2^k)}(X_{n-1}) \\ &= q^{k-1}\left(s_{(2^{k-1},1)}(X_{n-1}) s_{(2^k)}(X_m) - s_{(2^{k-1},1)}(X_m) s_{(2^k)}(X_{n-1})\right) \end{aligned}$$

$$\begin{aligned} & q^{2(n-1)} s_{(2^{k-1})}(X_{n-1}) s_{(2^k)}(X_m) - q^{2m} s_{(2^{k-1})}(X_m) s_{(2^k)}(X_{n-1}) \\ &= q^{2k(m+n-1)}\left(s_{(2^{k-1})}(X_{n-1}^{-1}) s_{(2^k)}(X_m^{-1}) - s_{(2^{k-1})}(X_m^{-1}) s_{(2^k)}(X_{n-1}^{-1})\right). \end{aligned}$$

q-Log-Concavity of $N_q(n,k)$ for fixed k

Theorem (Chen-Wang-Yang, J. Algebraic Combin. (2010))

Given an integer k, the sequence $(N_q(n,k))_{n\geq k}$ is strongly q-log-concave.

Proof. For any $m \geq n \geq k$,

$$N_q(m,k)N_q(n,k) - N_q(m+1,k)N_q(n-1,k).$$

$$\Downarrow$$

$$s_{(2^{k-1})}(X_m)s_{(2^{k-1})}(X_n) - s_{(2^{k-1})}(X_{m+1})s_{(2^{k-1})}(X_{n-1})$$

$$\Downarrow$$

$$s_{(2^{k-1})}(X_m)\left(s_{(2^{k-1})}(X_{n-1}) + q^{n-1}s_{(2^{k-2},1)}(X_{n-1}) + q^{2(n-1)}s_{(2^{k-2})}(X_{n-1})\right)$$

$$-\left(s_{(2^{k-1})}(X_m) + q^m s_{(2^{k-2},1)}(X_m) + q^{2m}s_{(2^{k-2})}(X_m)\right) s_{(2^{k-1})}(X_{n-1})$$

q-Log-Concavity of $N_q(n,k)$ for fixed k

$$\Downarrow$$

$$\left(q^{n-1}s_{(2^{k-2},1)}(X_{n-1})s_{(2^{k-1})}(X_m)-q^m s_{(2^{k-2},1)}(X_m)s_{(2^{k-1})}(X_{n-1})\right)$$

$$+\left(q^{2(n-1)}s_{(2^{k-2})}(X_{n-1})s_{(2^{k-1})}(X_m)-q^{2m}s_{(2^{k-2})}(X_m)s_{(2^{k-1})}(X_{n-1})\right).$$

$$\Downarrow \qquad \text{(Transformation formulas)}$$

$$q^{k-2}\left(s_{(2^{k-2},1)}(X_{n-1})s_{(2^{k-1})}(X_m)-s_{(2^{k-2},1)}(X_m)s_{(2^{k-1})}(X_{n-1})\right)$$

$$+q^{2(k-1)(m+n-1)}\left(s_{(2^{k-2})}(X_{n-1}^{-1})s_{(2^{k-1})}(X_m^{-1})-s_{(2^{k-2})}(X_m^{-1})s_{(2^{k-1})}(X_{n-1}^{-1})\right)$$

q-Log-Concavity of $N_q(n,k)$ for fixed k

Let $Z = X_m - X_{n-1}$, that is, $Z = \{q^{n-1}, \ldots, q^{m-1}\}$. Set $Z^{-1} = \{q^{1-n}, \ldots, q^{1-m}\}$. Then using

$$s_\lambda(X_m) = \sum_\mu s_\mu(X_{n-1}) s_{\lambda/\mu}(Z),$$

we have

$$q^{k-2} s_{(2^{k-2},1)}(X_{n-1}) s_{(2^{k-1})}(Z) + q^{2(k-1)(m+n-1)} s_{(2^{k-2})}(X_{n-1}^{-1}) s_{(2^{k-1})}(Z^{-1})$$

$$+ q^{k-2} \sum_{J \subseteq (2^{k-2},1)} s_J(Z) \left(s_{(2^{k-2},1)} s_{(2^{k-1})/J} - s_{(2^{k-2},1)/J} s_{(2^{k-1})} \right) (X_{n-1})$$

$$+ q^{2(k-1)(m+n-1)} s_{(2^{k-2})}(X_{n-1}^{-1}) s_{(2^{k-2},1)}(Z^{-1}) s_{(1)}(X_{n-1}^{-1})$$

$$+ q^{2(k-1)(m+n-1)} \sum_{I \subseteq (2^{k-2})} s_I(Z^{-1}) \left(s_{(2^{k-2})} s_{(2^{k-1})/I} - s_{(2^{k-2})/I} s_{(2^{k-1})} \right) (X_{n-1}^{-1}).$$

q-Log-Concavity of $N_q(n,k)$ for fixed k

Given two partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$, let

$$\begin{aligned} \lambda \vee \mu &= (\max(\lambda_1, \mu_1), \max(\lambda_2, \mu_2), \ldots), \\ \lambda \wedge \mu &= (\min(\lambda_1, \mu_1), \min(\lambda_2, \mu_2), \ldots). \end{aligned}$$

For two skew partitions λ/μ and ν/ρ, we define

$$\begin{aligned} (\lambda/\mu) \vee (\nu/\rho) &= (\lambda \vee \nu)/(\mu \vee \rho), \\ (\lambda/\mu) \wedge (\nu/\rho) &= (\lambda \wedge \nu)/(\mu \wedge \rho). \end{aligned}$$

Theorem (Lam-Postnikov-Pylyavaskyy, Amer. J. Math. (2007))

For any two skew partitions λ/μ and ν/ρ, the difference

$$s_{(\lambda/\mu)\vee(\nu/\rho)} s_{(\lambda/\mu)\wedge(\nu/\rho)} - s_{\lambda/\mu} s_{\nu/\rho}$$

is Schur positive.

q-Log-Concavity of $N_q(n,k)$ for fixed k

Corollary

Let k be a positive integer. If I, J are partitions with $I \subseteq (2^{k-1})$ and $J \subseteq (2^{k-1}, 1)$, then both

$$s_{(2^{k-1})}s_{(2^k)/I} - s_{(2^{k-1})/I}s_{(2^k)} \tag{23}$$

and

$$s_{(2^{k-1},1)}s_{(2^k)/J} - s_{(2^{k-1},1)/J}s_{(2^k)} \tag{24}$$

are Schur positive.

Proof. For (23), take $\lambda = (2^{k-1}), \mu = I, \nu = (2^k)$ and $\rho = \emptyset$. For (24), take $\lambda = (2^{k-1}, 1), \mu = J, \nu = (2^k)$ and $\rho = \emptyset$.
Remark. The q-Log-Concavity of $N_q(n,k)$ for fixed k follows from the above corollary.

Connection with a Conjecture of McNamara and Sagan

Define the operator $\mathcal{L}$ which maps a polynomial sequence $\{f_i(q)\}_{i\geq 0}$ to a polynomial sequence given by

$$\mathcal{L}(f_i(q)) := f_i(q)^2 - f_{i-1}(q)f_{i+1}(q).$$

A sequence $\{f_i(q)\}$ is k-fold q-log-concave if $\mathfrak{L}^j(f_i)$ is q-log-concave for $1 \leq j \leq k-1$.
If $\{f_i(q)\}$ is k-fold log-concave for any k, then it is said to be infinitely q-log-concave.

Conjecture (McNamara and Sagan, Adv. in Appl. Math. (2010))

For fixed k, the Gaussian polynomials $\begin{bmatrix} n \\ k \end{bmatrix}_{n\geq k}$ is infinitely q-log-concave.

Remark. For fixed n, they have shown that $\begin{bmatrix} n \\ k \end{bmatrix}_k$ is not 2-fold q-log-concave.

Connection with a Conjecture of McNamara and Sagan

For fixed k, subscript the $\mathcal{L}$-operator by n.

$$\mathfrak{L}_n\left(\begin{bmatrix} n \\ k \end{bmatrix}\right) = \frac{q^{n-k}}{[n]}\begin{bmatrix} n \\ k-1 \end{bmatrix}\begin{bmatrix} n \\ k \end{bmatrix},$$

which are, up to a power of q, the q-Narayana numbers.

$$\mathfrak{L}_n^2\left(\begin{bmatrix} n \\ k \end{bmatrix}\right) = \frac{q^{3n-3k}[2]}{[n]^2[n-1]}\begin{bmatrix} n \\ k \end{bmatrix}^2\begin{bmatrix} n \\ k-1 \end{bmatrix}\begin{bmatrix} n \\ k-2 \end{bmatrix}.$$

McNamara and Sagan (2010) conjectured that these polynomials are q-nonnegative.

Connection with a Conjecture of McNamara and Sagan

McNamara and Sagan (2010):
"It is not clear that these polynomials are q-nnonegative, although they must be if Conjecture 5.3 is true. Furthermore, when $q = 1$, the triangle made as n and k vary is not in Sloane's Encyclopedia [24] (although it has now been submitted). We expect that these integers and polynomials have interesting, yet to be discovered, properties."

Corollary (Chen-Wang-Yang, J. Algebraic Combin. (2010))

For fixed k, the Gaussian polynomials $\begin{bmatrix} n \\ k \end{bmatrix}_{n\geq k}$ *is 2-fold q-log-concave.*

Outline

q-log-convexity of Narayana polynonmials

Narayana polynomial of type A and B are defined respectively as follows:

$$NA_n(q) = \sum_{k=0}^{n} N(n,k)q^k,$$

and

$$NB_n(q) = \sum_{k=0}^{n} \binom{n}{k}^2 q^k.$$

Conjecture (Liu-Wang, Adv. in Appl. Math. (2007))

The polynomials $NA_n(q)$ form a q-log-convex sequence, so do $NB_n(q)$.

q-log-convexity of Narayana polynonmials

Theorem (Chen-Wang-Yang, J. Algebraic Combin. (2010))

The Narayana polynomials $NA_n(q)$ of type A are strongly q-log-convex.

Theorem (Chen-Wang-Yang, J. Algebraic Combin. (2010))

The Narayana polynomials $NB_n(q)$ of type B are q-log-convex.

Idea: q-log-convexity $\Rightarrow$ Schur positivity
Method: regard coefficients as specialization of symmetric functions.

Narayana polynonmials of type A

$$N(n,k) = N_q(n,k)|_{q=1} = s_{(2^{k-1})}(1^{n-1}) = \mathrm{ps}^1_{n-1}\left(s_{(2^{k-1})}\right).$$

$$[q^r]NA_{m+1}(q)NA_{n-1}(q) = \sum_{k=0}^{r-2} \mathrm{ps}^1_m\left(s_{(2^k)}\right)\mathrm{ps}^1_{n-2}\left(s_{(2^{r-2-k})}\right).$$

$$[q^r]NA_m(q)NA_n(q) = \sum_{k=0}^{r-2} \mathrm{ps}^1_{m-1}\left(s_{(2^k)}\right)\mathrm{ps}^1_{n-1}\left(s_{(2^{r-2-k})}\right)$$

Narayana polynonmials of type A

Given $a, b, m \in \mathbb{N}$ and $0 \leq i \leq m$, let

$$\begin{aligned} D_1(m,i,a,b) &= s_{(2^{i-b},1^{b-a})} s_{(2^{m-i-1})}, \\ D_2(m,i,a,b) &= s_{(2^{i-b-1},1^{b+2-a})} s_{(2^{m-i-1})}, \\ D_3(m,i,a,b) &= s_{(2^{i-b-1},1^{b+1-a})} s_{(2^{m-i-1},1)}, \end{aligned}$$

$$D(m,i,a,b) = D_1(m,i,a,b) + D_2(m,i,a,b) - D_3(m,i,a,b).$$

The coefficient $[q^r]\,(NA_{m+1}(q)NA_{n-1}(q) - NA_m(q)NA_n(q))$ is equal to

$$\mathrm{ps}^1_{n-2}\left(\sum_{0\leq a\leq b\leq d-1} \mathrm{ps}^1_d(s_{(2^a,1^{b+1-a})}) \sum_{k=0}^{r-2} D(r-2,k,a,b)\right).$$

Schur Positivity

Theorem (Chen-Wang-Yang, J. Algebraic Combin. (2010))

For any $b \geq a \geq 0$ and $m \geq 0$, the symmetric function $\sum_{i=0}^{m} D(m,i,a,b)$ is Schur positive.

Proof is based on the case of $a = b = 0$.

Given a set S of positive integers, let $\mathrm{Par}_S(n)$ denote the set of partitions of n whose parts belong to S.

Theorem (Chen-Wang-Yang, J. Algebraic Combin. (2010))

For any $m \geq 0$, we have

$$\sum_{i=0}^{m} D(m,i,0,0) = \sum_{\lambda \in \mathrm{Par}_{\{2,4\}}(2m-2)} s_\lambda. \tag{25}$$

Schur Positivity

Taking $m = 3, 4, 5$ and using the Maple package, we observe that

$$\sum_{k=0}^{3} \left(s_{(2^{k-1})} s_{(2^{3-k})} + s_{(2^{k-2},1^2)} s_{(2^{3-k})} - s_{(2^{k-1},1)} s_{(2^{3-k-1},1)} \right) = s_{(4)} + s_{(2,2)}.$$

$$\sum_{k=0}^{4} \left(s_{(2^{k-1})} s_{(2^{4-k})} + s_{(2^{k-2},1^2)} s_{(2^{4-k})} - s_{(2^{k-1},1)} s_{(2^{4-k-1},1)} \right) = s_{(4,2)} + s_{(2,2,2)}.$$

$$\sum_{k=0}^{5} \left(s_{(2^{k-1})} s_{(2^{5-k})} + s_{(2^{k-2},1^2)} s_{(2^{5-k})} - s_{(2^{k-1},1)} s_{(2^{5-k-1},1)} \right) = s_{(4,4)} + s_{(4,2,2)} + s_{(2,2,2,2)}.$$

The proof of the above theorem mainly relies on the recurrence relations of summands $D(m, i, 0, 0)$.

Experiment $\Rightarrow$ Observation $\Rightarrow$ Proof

An Operator Δ^μ

For a symmetric function f, suppose that f has the expansion $f = \sum_\lambda a_\lambda s_\lambda$, and then the action of Δ^μ on f is given by

$$\Delta^\mu(f) = \sum_\lambda a_\lambda s_{\lambda\cup\mu}.$$

Example

$$\begin{aligned} f &= s_{(4,3,2)} + 3s_{(2,2,1)} + 2s_{(5)} \\ \Delta^{(3,1)} f &= s_{(4,3,3,2,1)} + 3s_{(3,2,2,1,1)} + 2s_{(5,3,1)}. \end{aligned}$$

Some Identities of Symmetric Functions

For $a = b = 0$, denote $D(m, i, 0, 0)$ by $D_{m,i}$.

Theorem (Chen-Yang, J. Algebraic Combin. (2010))

Let $m = 2k + 1$ for some $k \in \mathbb{N}$.

(i) *We have*

$$\begin{aligned} D_{m,k} &= s_{(3^k)}s_{(1^k)}, \\ D_{m,k+1} &= s_{(4^k)} - s_{(3^k)}s_{(1^k)} - \Delta^{(2)}(s_{(3^k)}s_{1^{(k-2)}}). \end{aligned}$$

(ii) *For any $0 \leq i \leq k - 1$, we have*

$$\begin{aligned} D_{m,i} &= \Delta^{(2)}(D_{m-1,i}), \\ D_{m,m-i} &= \Delta^{(2)}(D_{m-1,m-1-i}). \end{aligned}$$

Some Identities of Symmetric Functions

Theorem

Let $m = 2k$ for some $k \in \mathbb{N}$.

(i) We have

$$\begin{aligned} D_{m,k-1} &= s_{(3^k)} s_{(1^{k-2})} + \Delta^{(2)}\big(s_{(3^{k-1})} s_{(1^{k-1})}\big), \\ D_{m,k} &= -s_{(3^k)} s_{(1^{k-2})}. \end{aligned}$$

(ii) For any $0 \le i \le k-2$, we have

$$\begin{aligned} D_{m,i} &= \Delta^{(2)}(D_{m-1,i}), \\ D_{m,m-i} &= \Delta^{(2)}(D_{m-1,m-1-i}), \\ D_{m,m-k+1} &= \Delta^{(2)}(D_{m-1,m-k}). \end{aligned}$$

$D_{m,k}$ for $m = 7$

	$m = 7$
$D_{7,0}$	$s_{(2^6)}$
$D_{7,1}$	$s_{(4,2^4)} + s_{(3^2,2^3)} + s_{(3,2^4,1)}$
$D_{7,2}$	$s_{(3^2,2^2,1^2)} + s_{(4,3^2,2)} + s_{(4^2,2^2)} + s_{(3^3,2,1)} + s_{(4,3,2^2,1)}$
$D_{7,3}$	$s_{(4,3^2,1^2)} + s_{(3^3,1^3)} + s_{(4^2,3,1)} + s_{(4^3)}$
$D_{7,4}$	$-s_{(4,3^2,2)} - s_{(4,3^2,1^2)} - s_{(3^3,2,1)} - s_{(3^3,1^3)} - s_{(4^2,3,1)}$
$D_{7,5}$	$-s_{(3^2,2^3)} - s_{(3^2,2^2,1^2)} - s_{(4,3,2^2,1)}$
$D_{7,6}$	$-s_{(3,2^4,1)}$
$D_{7,7}$	0

$D_{m,k}$ for $m = 8$

	$m = 8$
$D_{8,0}$	$s_{(2^7)}$
$D_{8,1}$	$s_{(4,2^5)} + s_{(3^2,2^4)} + s_{(3,2^5,1)}$
$D_{8,2}$	$s_{(3^2,2^3,1^2)} + s_{(4,3^2,2^2)} + s_{(4^2,2^3)} + s_{(3^3,2^2,1)} + s_{(4,3,2^3,1)}$
$D_{8,3}$	$s_{(4,3^2,2,1^2)} + s_{(3^3,2,1^3)} + s_{(4^2,3,2,1)} + s_{(4^3,2)}$ $+s_{(3^4,1^2)} + s_{(4^2,3^2)} + s_{(4,3^3,1)}$
$D_{8,4}$	$-s_{(3^4,1^2)} - s_{(4^2,3^2)} - s_{(4,3^3,1)}$
$D_{8,5}$	$-s_{(4^2,3,2,1)} - s_{(3^3,2^2,1)} - s_{(3^3,2,1^3)} - s_{(4,3^2,2,1^2)} - s_{(4,3^2,2^2)}$
$D_{8,6}$	$-s_{(3^2,2^4)} - s_{(3^2,2^3,1^2)} - s_{(4,3,2^3,1)}$
$D_{8,7}$	$-s_{(3,2^5,1)}$
$D_{8,8}$	0

Littlewood-Richardson Rule

Basic Tools for proving identities of symmetric functions: the
Littlewood-Richardson Rule
Littlewood-Richardson coefficients $c^{\lambda}_{\mu\nu}$:

$$s_\mu s_\nu = \sum_\lambda c^{\lambda}_{\mu\nu} s_\lambda.$$

Theorem

The Littlewood-Richardson coefficient $c^{\lambda}_{\mu\nu}$ is equal to the number of Littlewood-Richardson tableaux of shape λ/μ and type ν.

Littlewood-Richardson Rule

A lattice permutation of length n is a sequence $w_1 w_2 \cdots w_n$ such that for any i and j in the subsequence $w_1 w_2 \cdots w_j$ the number of i's is greater than or equal to the number of $i+1$'s.

The reverse reading word T^{rev} is a sequence of entries of T obtained by first reading each row from right to left and then concatenating the rows from top to bottom.

T is called a Littlewood-Richardson tableau,if the reverse reading word T^{rev} is a lattice permutation.

Littlewood-Richardson tableaux

Take $\lambda = (4,4,2,1)$, $\mu = (2,1)$, $\nu = (4,3,1)$. There are two Littlewood-Richardson tableaux of shape λ/μ and type ν ($c^{\lambda}_{\mu\nu} = 2$) as shown below.

*	*	1	1
*	1	2	2
1	3		
2			

*	*	1	1
*	1	2	2
1	2		
3			

Fig 3: Skew Littlewood-Richardson tableaux

Narayana polynomials of type B

When $\lambda = (1^k)$ for $k \geq 1$, the Schur function $s_\lambda(x)$ becomes the k-th elementary symmetric function $e_k(x)$, i.e.,

$$s_{(1^k)}(x) = e_k(x) = \sum_{1 \leq i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}. \tag{26}$$

$$NB_n(q) = \sum_{k=0}^{n} \binom{n}{k}^2 q^k.$$

$$[q^k](NB_n(q)) = \mathrm{ps}_n^1(e_k^2).$$

$$\mathrm{ps}_n^1(e_k) = \mathrm{ps}_{n-1}^1(e_k + e_{k-1}).$$

Narayana polynomials of type B

Reiner introduced the type B analogue of noncrossing partitions.

Theorem (Reiner, Discrete Math. (1997))

The lattice of non-crossing partitions of type B is a ranked self-dual lattice with cardinality $\binom{2n}{n}$ and rank generating function

$$NB_n(q) = \sum_{k=0}^{n} \binom{n}{k}^2 q^k.$$

Narayana polynomials of type B

The coefficient of q^r in $NB_{n-1}(q)NB_{n+1}(q) - (NB_n(q))^2$ is given by

$$\sum_{k=0}^{r} \mathrm{ps}_{n-1}^1(e_k)^2 \mathrm{ps}_{n+1}^1(e_{r-k})^2 - \mathrm{ps}_n^1(e_k)^2 \mathrm{ps}_n^1(e_{r-k})^2.$$

$\Downarrow$ apply $\mathrm{ps}_n^1(e_k) = \mathrm{ps}_{n-1}^1(e_k + e_{k-1})$ twice.

$$\mathrm{ps}_{n-1}^1 \left(\sum_{k=0}^{r} {e_k}^2 (e_{r-k} + 2e_{r-k-1} + e_{r-k-2})^2 - (e_k + e_{k-1})^2 (e_{r-k} + e_{r-k-1})^2 \right).$$

$$\Downarrow$$

$$2\,\mathrm{ps}_{n-1}^1 \left(\sum_{k=0}^{r} e_{k-1}^2 e_{r-k}^2 + e_{k-2} e_k e_{r-k}^2 - 2 e_{k-1} e_k e_{r-k-1} e_{r-k} \right).$$

Narayana polynomials of type B

Theorem (Chen-Tang-Wang-Yang, Adv. in Appl. Math. (2010))

For any $r \geq 1$, we have

$$\sum_{k=0}^{r}(e_{k-1}e_{k-1}e_{r-k}e_{r-k} + e_{k-2}e_k e_{r-k}e_{r-k} - 2e_{k-1}e_k e_{r-k-1}e_{r-k}) = \sum_{\lambda} s_\lambda,$$

where λ sums over all partitions of $2r-2$ of the form $(4^{i_4}, 3^{2i_3}, 2^{2i_2}, 1^{2i_1})$ with i_1, i_2, i_3, i_4 being nonnegative integers.

Remark. Proof mainly relies on the Jacobi-Trudi identity and the Pieri rule.

The Jacobi-Trudi Identity

Theorem

Let λ be a partition with the largest part $\leq n$ and λ' its conjugate. Then

$$s_\lambda(x) = \det(e_{\lambda_i{}' - i + j}(x))_{i,j=1}^{n},$$

where $e_0 = 1$ and $e_k = 0$ for $k < 0$.

$\sum_{k=0}^{r}(e_{k-1}e_{k-1}e_{r-k}e_{r-k} + e_{k-2}e_k e_{r-k}e_{r-k} - 2e_{k-1}e_k e_{r-k-1}e_{r-k})$ is equal to

$$\sum_{\substack{k=0 \\ k-1 \geq r-k}}^{r} e_{k-1}e_{r-k}s_{(2^{r-k},1^{2k-r-1})} - \sum_{\substack{k=0 \\ k-1 < r-k-1}}^{r} e_{k-1}e_{r-k}s_{(2^k,1^{r-2k-1})}$$

$$+ \sum_{\substack{k=0 \\ k-2 \geq r-k}}^{r} e_k e_{r-k}s_{(2^{r-k},1^{2k-r-2})} - \sum_{\substack{k=0 \\ k-2 < r-k-1}}^{r} e_k e_{r-k}s_{(2^{k-1},1^{r-2k})}.$$

The Pieri Rule

A skew partition λ/μ is called a horizontal (or vertical) strip if there are no two squares in the same column (resp. in the same row).

Theorem

We have $s_\mu s_{(n)} = \sum_\lambda s_\lambda$ summed over all partitions λ such that λ/μ is a horizontal strip of size n, and $s_\mu s_{(1^n)} = \sum_\lambda s_\lambda$ summed over all partitions λ such that λ/μ is a vertical strip of size n.

Outline

Polynomials with triangular recurrence relation

Consider

$$P_n(q) = \sum_{k=0}^{n} T(n,k)q^k, \quad n \geq 0,$$

where the coefficients $T(n,k)$ are nonnegative real numbers and satisfy the following recurrence relation

$$\begin{aligned} T(n,k) &= (a_1 n + a_2 k + a_3)T(n-1,k) \\ &\quad + (b_1 n + b_2 k + b_3)T(n-1,k-1), \text{ for } n \geq k \geq 1. \end{aligned} \tag{27}$$

Polynomials with triangular recurrence relation

We further need the boundary conditions

$$T(n,-1) = T(n,n+1) = 0, \text{ for } n \geq 1,$$

$$a_1 \geq 0, \quad a_1 + a_2 \geq 0, \quad a_1 + a_2 + a_3 > 0,$$

and

$$b_1 \geq 0, \quad b_1 + b_2 \geq 0, \quad b_1 + b_2 + b_3 > 0.$$

For the triangular array $\{T(n,k)\}_{n \geq k \geq 0}$, we always assume that $T(0,0) > 0$. Thus we have $T(n,k) > 0$ for $0 \leq k \leq n$.

Log-concavity of $T(n,k)$

Lemma (Kurtz, J. Combin. Theory Ser. A (1972))

Suppose that the positive array $\{T(n,k)\}_{n\geq k\geq 0}$ satisfies the recurrence relation (27). Then, for given n, the sequence $\{T(n,k)\}_{0\leq k\leq n}$ is log-concave, namely, for $0 \leq k \leq n$,

$$T(n,k)^2 \geq T(n,k-1)T(n,k+1).$$

A Sufficient Condition of q-Log-convexity

Theorem (Liu-Wang, Adv. in Appl. Math. (2007))

Suppose that the array $\{T(n,k)\}_{n\geq k\geq 0}$ *of positive numbers satisfies the recurrence relation* (27) *and the additional condition*

$$(a_2b_1 - a_1b_2)n + a_2b_2k + (a_2b_3 - a_3b_2) \geq 0, \quad \text{for } 0 < k \leq n.$$

Then the polynomials $P_n(q)$ *form a q-log-convex sequence.*

Remark. This theorem applies to the Bell polynomials and the Eulerian polynomials. Proof is based on Kurtz's result.

A key lemma

Lemma (Chen-Wang-Yang, Canad. Math. Bull., to appear)

Suppose that the array $\{T(n,k)\}_{n\geq k\geq 0}$ of positive numbers satisfies (27) with $a_2, b_2 \geq 0$. Then, for any $l' \geq l \geq 0$ and $m' \geq m \geq 0$, we have

$$T(m,l)T(m',l') - T(m,l')T(m',l) \geq 0.$$

In terms of polynomials, the lemma reads

$$P_n' P_{m-1} - P_n P_{m-1}' \geq_q 0.$$

Another Sufficient Condition of q-log-convexity

Note that

$$\begin{aligned} P_{m-1}P_{n+1} - P_mP_n &= (a_1 + b_1q)(n - m + 1)P_{m-1}P_n \\ &\quad + q(a_2 + b_2q)(P'_nP_{m-1} - P_nP'_{m-1}), \end{aligned}$$

Theorem (Chen-Wang-Yang, Canad. Math. Bull., to appear)

Suppose that the array $\{T(n,k)\}_{n\geq k\geq 0}$ of positive numbers satisfies (27) with $a_2, b_2 \geq 0$. Then the polynomial sequence $\{P_n(q)\}_{n\geq 0}$ is strongly q-log-convex.

Remark. This result applies to the Bell polynomials, the Bessel polynomials, the Ramanujan polynomials and the Dowling polynomials.

Another Sufficient Condition of q-log-convexity

The Ramanujan polynomials $R_n(x)$ are defined by the following recurrence relation:

$$R_1(x) = 1, \quad R_{n+1}(x) = n(1+x)R_n(x) + x^2 R_n'(x), \tag{28}$$

where $R_n'(x)$ is the derivative of $R_n(x)$ with respect to x. These polynomials are related to a refinement of Cayley's theorem due to Shor. Let $r(n,k)$ be the number of rooted labeled trees on n vertices with k improper edges. Shor proved that $R_n(x)$ is the generating function of $r(n,k)$.

Let $r'(n,k) = r(n+1,k)$. Then the triangle $\{r'(n,k)\}_{n \geq k \geq 0}$ satisfies the following recurrence relation

$$r'(n,k) = nr'(n-1,k) + (n+k-1)r'(n-1,k-1), \tag{29}$$

which leads to the q-log-convexity of $R_n(x)$.

Outline

Longest Increasing Subsequences

Let

$$P_n(q) = \sum_k P_{n,k} q^k,$$

where $P_{n,k}$ is the number of permutations π on $[n] = \{1, 2, \ldots, n\}$ such that the length of the longest increasing subsequences of π equals k.

Theorem (Baik-Deift-Johansson, J. Amer. Math. Soc. (1999))

The limiting distribution of the coefficients of $P_n(q)$ is the Tracy-Widom distribution.

The numbers $P_{n,k}$ can be computed by Gessel's theorem. Let $\mathfrak{S}_n$ be the symmetric group on $[n]$, and let $\mathrm{is}(\pi)$ be the length of the longest increasing subsequences of π.

Longest Increasing Subsequences

Define

$$u_k(n) = \sharp\{w \in \mathfrak{S}_n : \mathrm{is}(w) \leq k\}, \tag{30}$$

$$U_k(q) = \sum_{n\geq 0} u_k(n) \frac{q^{2n}}{n!^2}, \quad k \geq 1, \tag{31}$$

$$I_i(2q) = \sum_{n\geq 0} \frac{q^{2n+i}}{n!(n+i)!}, \quad i \in \mathbb{Z}. \tag{32}$$

Theorem (Gessel, J. Combin. Theory, Ser. A (1990))

$$U_k(q) = \det(I_{i-j}(2q))_{i,j=1}^{k}.$$

Longest Increasing Subsequences

Note that $P_{n,k} = u_k(n) - u_{k-1}(n)$ for $n \geq 1$.

$$P_1(q) = q,$$

$$P_2(q) = q + q^2,$$

$$P_3(q) = q + 4q^2 + q^3,$$

$$P_4(q) = q + 13q^2 + 9q^3 + q^4,$$

$$P_5(q) = q + 41q^2 + 61q^3 + 16q^4 + q^5,$$

$$P_6(q) = q + 131q^2 + 381q^3 + 181q^4 + 25q^5 + q^6,$$

$$P_7(q) = q + 428q^2 + 2332q^3 + 1821q^4 + 421q^5 + 36q^6 + q^7.$$

Longest Increasing Subsequences

Conjecture

$P_n(q)$ is log-concave for $n \geq 1$.

Conjecture

$P_n(q)$ is ∞-log-concave for $n \geq 1$.

Conjecture

The polynomial sequence $\{P_n(q)\}$ is strongly q-log-convex.

Conjecture

The polynomial sequence $\{P_n(q)\}$ is infinitely q-log-convex.

Longest Increasing Subsequences

Let $f^{\lambda/\mu}$ denote the number of standard Young tableaux of shape λ/μ. The exponential specialization is a homomorphism $ex : \Lambda \to \mathbb{Q}[t]$, defined by $ex(p_n) = t\delta_{1n}$, where p_n is the n-th power sum. Let $ex_1(f) = ex(f)_{t=1}$, provided this number is defined. It is known that

$$ex_1(s_{\lambda/\mu}) = \frac{f^{\lambda/\mu}}{|\lambda/\mu|!}, \quad P_{n,k} \overset{RSK}{=} \sum_{\lambda \vdash n, \lambda_1 = k} \left(f^{\lambda}\right)^2.$$

Conjecture

Let

$$f_{n,k} = \sum_{\lambda \vdash n, \lambda_1 = k} s_\lambda^2.$$

Then $f_{n,k}^2 - f_{n,k+1} f_{n,k-1}$ *is s-positive for* $1 \le k \le n$.

Remark. This conjecture implies the log-concavity of $P_{n,k}$.

Matchings with Given Crossing Number

Let

$$M_{2n}(q) = \sum_k M_{2n,k} q^k,$$

where $M_{2n,k}$ is the number of matchings on $[2n]$ with crossing number k. Let

$$V_k(q) = \sum_{n \geq 0} v_k(n) \frac{q^n}{n!},$$

where $v_k(n)$ denotes the number of matchings on $[2n]$ whose crossing number is less than or equal to k.

Theorem (Grabiner-Magyar, J. Algebraic Combin. (1993); Goulden, Discrete Math. (1992))

$$V_k(q) = \det(I_{i-j}(2q) - I_{i+j}(2q))_{i,j=1}^{k}.$$

Matchings with Given Crossing Number

Note that $M_{2n,k} = v_k(n) - v_{k-1}(n)$.

$$M_2(q) = q$$

$$M_4(q) = 2q + q^2$$

$$M_6(q) = 5q + 9q^2 + q^3$$

$$M_8(q) = 14q + 70q^2 + 20q^3 + q^4$$

$$M_{10}(q) = 42q + 552q^2 + 315q^3 + 35q^4 + q^5$$

$$M_{12}(q) = 132q + 4587q^2 + 4730q^3 + 891q^4 + 54q^5 + q^6$$

$$M_{14}(q) = 429q + 40469q^2 + 71500q^3 + 20657q^4 + 2002q^5 + 77q^6 + q^7$$

Matchings with Given Crossing Number

Conjecture

$M_{2n}(q)$ is log-concave for $n \geq 1$.

Conjecture

$M_{2n}(q)$ is ∞-log-concave for $n \geq 1$.

Conjecture

The polynomial sequence $\{M_{2n}(q)\}$ is strongly q-log-convex.

Conjecture

The polynomial sequence $\{M_{2n}(q)\}$ is infinitely q-log-concavity.

Integer Partition

Let p_n be the number of partitions of n. The first numbers of p_n are stated as follow:

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176,$$

$$231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, 2436,$$

$$3010, 3718, 4565, 5604, 6842, 8349, 10143, 12310,$$

Integer Partition

We propose the following conjectures.

Conjecture

The sequence $\{p_n\}_{n\geq 26}$ is log-concave. In other words, for any $n \geq 26$,

$$\frac{p_{n-1}}{p_n} < \frac{p_n}{p_{n+1}}.$$

The truth can be verified for $n \leq 8000$.

Integer Partition

Conjecture

For any $m \geq 2$ and any $n \geq m+1$, we have

$$\frac{p_{n-m}}{p_n} < \frac{p_n}{p_{n+m}}. \tag{33}$$

In particular, the case $m = 2$ states that both the sequences $\{p_{2n}\}_{n\geq 1}$ and $\{p_{2n-1}\}_{n\geq 1}$ are log-concave.

Integer Partition

Conjecture

For any constants $a > b$ and any $n \geq 1$,

$$\frac{p_{(a-b)n}}{p_{an}} < \frac{p_{an}}{p_{(a+b)n}}.$$

Conjecture

For any $n \geq 1$,

$$\frac{p_{n-1}}{p_n}\left(1+\frac{1}{n}\right) > \frac{p_n}{p_{n+1}}.$$

Conjecture

For all k, there exists a constant $n_0(k)$ such that $P(n)$ is k-log-concave for $n > n_0(k)$.

An Asymptotical Result

Theorem (Canfield, 1995)

$$p_n = \frac{e^{c\sqrt{n}}}{4\sqrt{3}n}\left(1 + \frac{c_1}{\sqrt{n}} + \frac{c_2}{n} + \frac{c_3}{n^{3/2}} + O(n^{-2})\right),$$

where

$$c = \sqrt{\frac{2}{3}}\,\pi.$$

In other words, there exists d and n_0 such that when $n > n_0$,

$$\frac{e^{c\sqrt{n}}}{4\sqrt{3}n}\left(1+\frac{c_1}{\sqrt{n}}+\frac{c_2}{n}+\frac{c_3}{n^{3/2}}-dn^{-2}\right) < p_n < \frac{e^{c\sqrt{n}}}{4\sqrt{3}n}\left(1+\frac{c_1}{\sqrt{n}}+\frac{c_2}{n}+\frac{c_3}{n^{3/2}}+dn^{-2}\right),$$

Remark. This implies the log-concavity of p_n for sufficiently large n. However, we can not give an exact value n_0 such that when $n > n_0$ the log-concavity holds.

q-Catalan Number

The Catalan numbers:

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The usual q-analog of the Catalan numbers is given by

$$C_n(q) := \frac{1}{[n+1]}\begin{bmatrix}2n\\ n\end{bmatrix}.$$

Let

$$m_n(i) = [q^i]C_n(q).$$

q-Catalan Number

We use the moment generating function technique to obtain the following result.

Theorem (Chen-Wang-Wang, Proc. Amer. Math. Soc. (2008))

The limiting distribution of the coefficients of the q-Catalan numbers is normal.

By a similar argument, we obtain two general theorems.

Corollary (Chen-Wang-Wang, Proc. Amer. Math. Soc. (2008))

The distribution of the coefficients in $c_n(q) = \frac{[2]}{[2n]}\begin{bmatrix} 2n \\ n-1 \end{bmatrix}$ is asymptotically normal.

Corollary (Chen-Wang-Wang, Proc. Amer. Math. Soc. (2008))

The coefficients of the generalized q-Catalan numbers $C_{n,m}(q) = \frac{1}{[(m-1)n+1]}\begin{bmatrix} mn \\ n \end{bmatrix}$ are normally distributed when $n \to \infty$.

q-Catalan Number

The coefficients of q-Catalan number is not unimodal, since it is obvious that the coefficient of q in $C_n(q)$ is zero, see Stanley (1989). For example,

$$C_4(q) = 1 + q^2 + q^3 + 2q^4 + q^5 + 2q^6 + q^7 + 2q^8 + q^9 + q^{10} + q^{12}.$$

Conjecture (Chen-Wang-Wang, Proc. Amer. Math. Soc. (2008))

The sequence $\{m_n(1), \ldots, m_n(n(n-1)-1)\}$ is unimodal when n is sufficiently large.

Conjecture (Chen-Wang-Wang, Proc. Amer. Math. Soc. (2008))

There exists an integer t such that when n is sufficiently large, the sequence $\{m_n(t), \ldots, m_n(n(n-1)-t)\}$ is log-concave.

Faulhaber polynomials

Setting $u = n^2 + n$, Faulhaber coefficient $A_k^{(m)}$ be defined by

$$\sum_{i=1}^{n} i^{2m-1} = \frac{1}{2m} \sum_{k=0}^{m} A_k^{(m)} u^{m-k}.$$

Theorem (Knuth, Math. Comput. (1993))

$$A_0^{(m)} = 1, \quad \sum_{j=0}^{k} \binom{m-j}{2k+1-2j} A_j^{(m)} = 0, \quad k \geq 0.$$

Theorem (Knuth, Math. Comput. (1993))

Faulhaber coefficients satisfies the following recurrence relation

$$(2m-2k)(2m-2k-1)A_k^{(m)} + (m-k+1)(m-k)A_{k-1}^{(m)} = 2m(2m-1)A_k^{(m-1)}.$$

Faulhaber polynomials

Conjecture

The sequence $\{|A_k^{(m)}|\}_{0\le k\le m-2}$ *is log-concave.*

Example

Setting $N = n(n+1)/2$, *we have*

$$1 + 2 + \cdots + n = N$$

$$1^3 + 2^3 + \cdots + n^3 = N^2$$

$$1^5 + 2^5 + \cdots + n^5 = (4N^3 - N^2)/3$$

$$1^7 + 2^7 + \cdots + n^7 = (12N^4 - 8N^3 + 2N^2)/6$$

$$1^9 + 2^9 + \cdots + n^9 = (16N^5 - 20N^4 + 12N^3 - 3N^2)/5$$

$$1^{11} + 2^{11} + \cdots + n^{11} = (32N^6 - 64N^5 + 68N^4 - 40N^3 + 10N^2)/6$$

The End

Thank you!!!